International Journal of Theoretical Physics, Vol. 28, No. 11, 1989

Relativistic Zeeman Effect in Positronium, $n = 2$

Andrea Raspini[1]

Received February 5, 1989; revised May 1, 1989

Approximate relativistic center-of-mass variables are employed for the calculation of the relativistic corrections to the Zeeman structure of the $n = 2$ energy level of positronium.

1. INTRODUCTION AND NOTATION

This paper calculates the relativistic corrections to the Zeeman structure of the $n = 2$ energy level of positronium. Even though the topic is not new, the method is very advantageous, based on the use of relativistic center-of-mass (CM) variables (Krajcik and Foldy, 1974), as previously discussed by the author in a more detailed work on the Zeeman effect (Raspini, 1985). As a consequence of the consistency of the procedure, results may be considered quite reliable at this order of approximation (v^2/c^2) and in first-order perturbation theory.

In Section 2 the reduced Breit Hamiltonian is introduced (Breit, 1947; Barker and Glover, 1955; Close and Osborn, 1970; Krajcik and Foldy, 1974), which is correct up to the order of v^2/c^2; the "annihilation term" can also be included for the case of a particle-antiparticle system (Karplus and Klein, 1952). In the same section, the external electromagnetic (EM) interaction is generated (Sebastian, 1981; Raspini, 1985) for a uniform, constant, and weak magnetic field. To maintain, in a simple way, the consistency in the order of approximation, relativistic CM variables are adopted (Krajcik and Foldy, 1974). The elimination of the kinetic whole-body effects is then achieved by considering (unperturbed) rest frame states (Raspini, 1985): these are the usual zero-momentum bound eigenstates of the nonrelativistic isolated Hamiltonian. In Section 3 the perturbation problem is examined for positronium or other particle–antiparticle systems

[1]Department of Physics, SUNY at Fredonia, Fredonia, New York 14063.

0020-7748/89/1100-1359$06.00/0

bound by internal EM interaction; the results are presented in Sections 4 and 5. For a more complete discussion of the method see Raspini (1985) and references quoted therein.

Notation is standard throughout the paper, with one exception: if a function (or operator) is first introduced as $f(x)$, the symbols f or $f(x)$ will indicate the value of the function as originally defined; $\hat{f}$ will be used, instead, to signify the functional form. Therefore, in the case of a replacement $x \to y$, I will write $\hat{f}(y)$ rather than $f(y)$ [compare equations (1) and (4) in the following].

2. A REVIEW OF THE METHOD

For an isolated system of two spin-1/2 fermions bound by internal EM interaction, one can propose the use of a reduced Breit Hamiltonian (Krajcik and Foldy, 1974):

$$H_B(\mathbf{p}_k, \mathbf{r}_k, \mathbf{S}_k) = H_B^{(0)}(\mathbf{p}_k, \mathbf{r}_k) + H_B^{(1)}(\mathbf{p}_k, \mathbf{r}_k, \mathbf{S}_k), \qquad k = 1, 2 \tag{1}$$

where $H_B^{(0)}$ contains the usual nonrelativistic terms

$$H_B^{(0)} = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + \frac{1}{2} \sum_{i \neq j} \frac{K_{ij}}{r_{ij}}, \qquad i, j = 1, 2 \tag{2}$$

$$\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j, \qquad r_{ij} = |\mathbf{r}_{ij}|, \qquad K_{ij} = e_i e_j \tag{3}$$

and $H_B^{(1)}$ represents the correction to $H_B^{(0)}$, of relative order $1/c^2$. [For an explicit expression of $H_B^{(1)}$ see Raspini (1985).] The employed symbols are standard: $\mathbf{p}_i = \{p_i^b\}$, $\mathbf{r}_i = \{r_i^b\}$, and $\mathbf{S}_i = \{S_i^b\}$ (with $b = x, y, z$), denote the canonical momentum operator, position, and spin-1/2 operator of the particle labeled i, in an inertial, Cartesian, orthogonal, right-handed frame of reference $C(x, y, z)$; m_i and e_i indicate masses and charges.

In the presence of an external uniform and constant (weak) magnetic field $\mathbf{B}$, the total Hamiltonian can be obtained as follows (Sebastian, 1981; Raspini, 1985):

$$H_{\text{tot}} = \hat{H}_B(\mathbf{p}_k - e_k \mathbf{A}_k, \mathbf{r}_k, \mathbf{S}_k) + \mathcal{H} \tag{4}$$

$$\mathbf{A}_k = \tfrac{1}{2}\mathbf{B} \times \mathbf{r}_k, \qquad \mathcal{H} = \mathcal{H}^{(0)} + \mathcal{H}^{(1)} \tag{5}$$

$$\mathcal{H}^{(0)} = -\mathbf{B} \cdot \left[\sum_i \frac{e_i(1 + a_i)}{m_i} \mathbf{S}_i \right] \tag{6}$$

$$\mathcal{H}^{(1)} = \mathbf{B} \cdot \left\{ \sum_i \frac{e_i}{2m_i^3} [\mathbf{p}_i^2 \mathbf{S}_i + a_i (\mathbf{p}_i \cdot \mathbf{S}_i) \mathbf{p}_i] \right\} \tag{7}$$

The given $\mathcal{H}^{(0)}$ (nonrelativistic) and $\mathcal{H}^{(1)}$ (relative order $1/c^2$) neglect nonlinear field contributions (weak $\mathbf{B}$ field); the anomalous magnetic moments

a_i are taken into account, as they are in $H_B^{(1)}$ (Raspini, 1985). Also note that we are working with units in which $\hbar = c = 1$. Defining the interaction Hamiltonian $H_I = H_{\text{tot}} - H_B$, one can easily separate it into two parts (Raspini, 1985):

$$H_I = H_I^{(0)} + H_I^{(1)} \tag{8}$$

where $H_I^{(0)}$ is nonrelativistic

$$H_I^{(0)} = -\mathbf{B} \cdot \left(\sum_i \frac{e_i}{2m_i} \mathbf{L}_i \right) + \mathscr{H}^{(0)} \tag{9}$$

$$\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i \tag{10}$$

and $H_I^{(1)}$ is of relative order $1/c^2$: both are calculated keeping only linear terms in the external interaction. [For an explicit expression of $H_I^{(1)}$ see Raspini (1985).]

The definition of internal momenta ($\boldsymbol{\pi}_i$), positions ($\boldsymbol{\rho}_i$), and spin operators ($\boldsymbol{\sigma}_i$), and that of total momentum ($\mathbf{P} = \sum_i \mathbf{p}_i$) and position ($\mathbf{R}$), can be done at the order of approximation we are working with ("relativistic CM variables"). According to Krajcik and Foldy (1974),

$$\mathbf{p}_i = [\boldsymbol{\pi}_i + (m_i/M)\mathbf{P}] + \mathbf{F}_i(\text{CM variables}), \qquad M = \sum_i m_i \tag{11}$$

$$\mathbf{r}_i = [\boldsymbol{\rho}_i + \mathbf{R}] + \mathbf{G}_i(\text{CM variables}) \tag{12}$$

$$\mathbf{S}_i = [\boldsymbol{\sigma}_i] + \mathbf{H}_i(\text{CM variables}) \tag{13}$$

$$\sum_i \boldsymbol{\pi}_i = \mathbf{0}, \qquad \sum_i m_i \boldsymbol{\rho}_i = \mathbf{0} \tag{14}$$

In equations (11)-(13), $\mathbf{F}_i$, $\mathbf{G}_i$, and $\mathbf{H}_i$ represent appropriate corrections, of order $1/c^2$ relative to the expressions in square brackets. [For more details see Krajcik and Foldy (1974), Sebastian (1979), Sebastian and Yun (1979), and Raspini (1985).] When using the replacements (11)-(14), the corrections are to be included for expressing $H_B^{(0)}$ and $H_I^{(0)}$ in terms of CM variables; $H_B^{(1)}$ and $H_I^{(1)}$ simply need the terms in square brackets (Raspini, 1985): this is consistent with the overall order of approximation. After the algebra is done, we obtain the following results (Raspini, 1985):

$$H_B(\mathbf{p}_k, \mathbf{r}_k, \mathbf{S}_k) = h_B^{(0)}(\boldsymbol{\pi}_k, \boldsymbol{\rho}_k) + h_B^{(1)}(\boldsymbol{\pi}_k, \boldsymbol{\rho}_k, \boldsymbol{\sigma}_k) + O_B \tag{15}$$

$$H_I(\mathbf{p}_k, \mathbf{r}_k, \mathbf{S}_k) = h_I^{(0)}(\boldsymbol{\pi}_k, \boldsymbol{\rho}_k, \boldsymbol{\sigma}_k) + h_I^{(1)}(\boldsymbol{\pi}_k, \boldsymbol{\rho}_k, \boldsymbol{\sigma}_k) + O_I \tag{16}$$

where the meanings of O_B and O_I are given by

$$O_B |\Psi_{\text{cm}}\rangle = 0 \tag{17}$$

$$\langle \Psi_{\text{cm}} | O_I | \Psi'_{\text{cm}} \rangle = 0 \tag{18}$$

if $|\Psi_{cm}\rangle$ and $|\Psi'_{cm}\rangle$ are state vectors such that

$$\mathbf{P}|\Psi_{cm}\rangle = \mathbf{P}|\Psi'_{cm}\rangle = \mathbf{0} \tag{19}$$

The forms of the operators $h_B^{(0)}$, $h_I^{(0)}$, $h_B^{(1)}$, and $h_I^{(1)}$ are examined in Raspini (1985); the same operators will also be listed in equations (22) and (23) for the case of positronium.

Since we are dealing with two-particle systems, it is to be remarked that the whole space can be spanned by the independent coordinates $\mathbf{R}$ (canonically conjugated to $\mathbf{P}$) and $\boldsymbol{\rho} = \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$ (canonically conjugated to $\boldsymbol{\pi} = \boldsymbol{\pi}_1 = -\boldsymbol{\pi}_2$). In terms of these variables, the volume element may be defined in the form

$$dv = d\mathbf{R}\, d\boldsymbol{\rho} \tag{20}$$

$$\int d\mathbf{R} = D^3, \qquad D \to \infty \tag{21}$$

For the simple case of a particle–antiparticle system (e.g., positronium), the explicit expressions of $h_B^{(0)}$, $h_B^{(1)}$, $h_I^{(0)}$, and $h_I^{(1)}$ are as follows (Raspini, 1985):

$$h_B^{(0)} = \frac{\boldsymbol{\pi}^2}{m} - \frac{\alpha}{\rho} \tag{22}$$

$$h_I^{(0)} = -\frac{e(1+a)}{m}(\mathbf{B}\cdot\boldsymbol{\tau}) \tag{23}$$

$$\begin{aligned} h_B^{(1)} = {} & -\frac{\boldsymbol{\pi}^4}{4m^3} - \frac{\alpha}{2m^2}\left(\boldsymbol{\pi}\frac{1}{\rho}\cdot\boldsymbol{\pi}\right) - \frac{\alpha}{2m^2}(\boldsymbol{\pi}\cdot\boldsymbol{\rho})\frac{1}{\rho^3}(\boldsymbol{\rho}\cdot\boldsymbol{\pi}) \\ & + \frac{\alpha(3+4a)}{2m^2}\left[\boldsymbol{\sigma}\cdot\left(\frac{\boldsymbol{\rho}}{\rho^3}\times\boldsymbol{\pi}\right)\right] \\ & + \frac{\alpha\pi}{m^2}\left[-(1+2a+2a^2) + \frac{4(1+a)^2+3}{3}\boldsymbol{\sigma}^2\right]\delta(\boldsymbol{\rho}) \\ & - \frac{\alpha(1+a)^2}{2m^2}\frac{1}{\rho^3}\left[\boldsymbol{\sigma}^2 - 3\frac{(\boldsymbol{\sigma}\cdot\boldsymbol{\rho})^2}{\rho^2}\right] \end{aligned} \tag{24}$$

$$\begin{aligned} h_I^{(1)} = {} & \frac{3e}{4m^3}\boldsymbol{\pi}^2(\mathbf{B}\cdot\boldsymbol{\tau}) - \frac{e(1-2a)}{4m^3}(\boldsymbol{\pi}\cdot\boldsymbol{\tau})(\mathbf{B}\cdot\boldsymbol{\pi}) \\ & + \frac{\alpha e}{8m^2}\frac{1}{\rho}(\mathbf{B}\cdot\boldsymbol{\tau}) - \frac{\alpha e}{8m^2}\frac{1}{\rho^3}(\boldsymbol{\rho}\cdot\boldsymbol{\tau})(\mathbf{B}\cdot\boldsymbol{\rho}) \end{aligned} \tag{25}$$

$$m = m_1 = m_2, \qquad e = e_1 = -e_2, \qquad \alpha = e^2, \qquad a = a_1 = a_2 \tag{26}$$

$$\rho = |\boldsymbol{\rho}|, \qquad \boldsymbol{\tau} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 \tag{27}$$

The "annihilation" potential (Karplus and Klein, 1952; Sakurai, 1967; Grotch and Kashuba, 1973) is included in the fifth term of equation (24). In the same equation, the relationship $\boldsymbol{\sigma}_i^2 = 3/4$ has been utilized whenever convenient.

For pure Dirac particles, which is certainly the case here, a is of the order of α (and $\alpha \simeq 1/137$). The energy corrections obtained from (24) [see equation (38) for clarity] are of magnitude $m\alpha^4$, with contributions of higher order in α due exclusively to the presence of the aforementioned anomalous moments: other $m\alpha^5$ corrections cannot be predicted by means of (24). See, for instance, Barker and Glover (1955) for a Hamiltonian similar to ours; also see Fulton and Martin (1954) for a quantum electrodynamics calculation of all the $m\alpha^5$ energy corrections to the $n=2$ level of positronium.

3. THE PERTURBATION PROBLEM

The unperturbed states will be chosen to be eigenvectors of $\mathbf{P}$, corresponding to the eigenvalue $\mathbf{0}$. Furthermore, they will be specified as (bound) eigenstates of the isolated nonrelativistic Hamiltonian $H_B^{(0)}(\mathbf{p}_k, \mathbf{r}_k)$, of the orbital angular momentum $\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i$, of the spin $\mathbf{S} = \sum_i \mathbf{S}_i$, and of the total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$. (We intend that they will be eigenstates of $H_B^{(0)}$, $\mathbf{L}^2$, $\mathbf{S}^2$, $\mathbf{J}^2$, and J^z.) This results in the equations (Raspini, 1985)

$$\mathbf{P}|\cdot\rangle = \mathbf{0} \tag{28}$$

$$\left(\frac{\boldsymbol{\pi}^2}{m} - \frac{\alpha}{\rho}\right)|\cdot\rangle = E(n)|\cdot\rangle, \qquad E(n) = -\frac{m\alpha^2}{4n^2}, \qquad n = 1, 2, \ldots \tag{29}$$

$$(\boldsymbol{\rho} \times \boldsymbol{\pi})^2|\cdot\rangle = l(l+1)|\cdot\rangle, \qquad l = 0, \ldots, n-1 \tag{30}$$

$$\boldsymbol{\sigma}^2|\cdot\rangle = s(s+1)|\cdot\rangle, \qquad s = 0, 1 \tag{31}$$

$$(\boldsymbol{\rho} \times \boldsymbol{\pi} + \boldsymbol{\sigma})^2|\cdot\rangle = j(j+1)|\cdot\rangle, \qquad j = |l-s|, \ldots, l+s \tag{32}$$

$$[(\rho^x \pi^y - \rho^y \pi^x) + \sigma^z]|\cdot\rangle = j^z|\cdot\rangle, \qquad j^z = -j, \ldots, j \tag{33}$$

The (bound) eigensolutions of equations (28)–(33) are well known (Messiah, 1966), and may be labeled as

$$|\mathbf{0}, n, l, s, j, j^z\rangle \tag{34}$$

The perturbed eigenvalue problem is

$$(H_B + H_I)|\Psi\rangle = E|\Psi\rangle \tag{35}$$

By means of first-order (degenerate) perturbation theory, one obtains the secular equation for the nth level (Raspini, 1985)

$$\det[\langle \mathbf{0}, n, \gamma | h_B^{(1)} + h_I^{(0)} + h_I^{(1)} | \mathbf{0}, n, \gamma' \rangle - \varepsilon(n)\delta_{\gamma\gamma'}] = 0 \tag{36}$$

where γ, γ' number the degenerate states in the level, and $\varepsilon(n)$ is the energy correction to $E(n)$. The case $n=1$ was examined in a previous paper (Raspini, 1985): here I study the structure of the $n=2$ level. To that end, note that

$$\langle \mathbf{0}, n=2, \gamma | h_B^{(1)} | \mathbf{0}, n=2, \gamma' \rangle = \delta_{\gamma\gamma'} W(n=2, l, s, j) \tag{37}$$

where

$$W(n, l, s, j) = \langle \mathbf{0}, n, l, s, j, j^z | h_B^{(1)} | \mathbf{0}, n, l, s, j, j^z \rangle \tag{38}$$

The γ, γ' indices in equations (36) and (37) can be assigned as follows:

$$\begin{aligned}
&l=0, \quad s=0, \quad j=0, \quad j^z=0 &&\Rightarrow \quad \gamma=1\\
&l=0, \quad s=1, \quad j=1, \quad j^z=-1,0,1 &&\Rightarrow \quad \gamma=2,3,4\\
&l=1, \quad s=0, \quad j=1, \quad j^z=-1,0,1 &&\Rightarrow \quad \gamma=5,6,7\\
&l=1, \quad s=1, \quad j=0, \quad j^z=0 &&\Rightarrow \quad \gamma=8\\
&l=1, \quad s=1, \quad j=1, \quad j^z=-1,0,1 &&\Rightarrow \quad \gamma=9,10,11\\
&l=1, \quad s=1, \quad j=2, \quad j^z=-2,-1,0,1,2 &&\Rightarrow \quad \gamma=12,13,14,15,16
\end{aligned}$$

The appropriate W values are available in the scientific literature[2] and will not be repeated here. Equation (37) is justified by the fact that $h_B^{(1)}$ is even under the spatial parity inversion $\boldsymbol{\pi} \to -\boldsymbol{\pi}$, $\boldsymbol{\rho} \to -\boldsymbol{\rho}$ (while states with $l=0$ and $l=1$ have opposite partities), and commutes with the operators in equations (31)-(33).

4. ZEEMAN STRUCTURE OF THE $n=2$ ENERGY LEVEL OF POSITRONIUM

The secular equation can be rewritten in the convenient form

$$\det[T_{\gamma\gamma'} - \lambda_\gamma \delta_{\gamma\gamma'}] = 0, \qquad \gamma, \gamma' = 1, \ldots, 16 \tag{39}$$

$$T_{\gamma\gamma'} = \langle \mathbf{0}, n=2, \gamma | h_I | \mathbf{0}, n=2, \gamma' \rangle, \qquad h_I = h_I^{(0)} + h_I^{(1)} \tag{40}$$

$$\lambda_\gamma = \varepsilon(2) - W_\gamma, \qquad W_\gamma = W(n=2, l, s, j) \tag{41}$$

[2]See, for instance, Grotch and Kashuba (1973) for $m\alpha^4$ values; also see Barker and Glover (1955).

where, according to the assignment of the γ indices, $W_2 = W_3 = W_4$; $W_5 = W_6 = W_7$; $W_9 = W_{10} = W_{11}$; $W_{12} = W_{13} = \cdots = W_{16}$. Noticing that h_I is even under spatial parity, one obtains the following structure for T:

$$T = \left(\begin{array}{c|c} U & N' \\ \hline N'' & V \end{array}\right) \tag{42}$$

in which U is 4×4 (h_I matrix elements of the $l=0$ states), V is 12×12 (h_I matrix elements of the $l=1$ states), and N', N'' are null. Therefore

$$\det[U_{\gamma\gamma'} - \lambda_\gamma \delta_{\gamma\gamma'}] = 0, \qquad \gamma, \gamma' = 1, \ldots, 4 \tag{43}$$

$$\det[V_{\gamma\gamma'} - \lambda_\gamma \delta_{\gamma\gamma'}] = 0, \qquad \gamma, \gamma' = 5, \ldots, 16 \tag{44}$$

are equivalent to equation (39).

Examining equations (23), (25), it follows that all matrix elements of h_I between states with the same s must vanish, since h_I is odd under the spin interchange $\boldsymbol{\sigma}_1 \leftrightarrow \boldsymbol{\sigma}_2$ (while $s=0$ states are odd, and $s=1$ states are even); clearly, h_I is also odd under the time reversal: $\boldsymbol{\pi} \to -\boldsymbol{\pi}$, $\boldsymbol{\sigma}_1 \to -\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2 \to -\boldsymbol{\sigma}_2$. With $B^x = B^y = 0$, $B^z = B$ (which shall be enforced from now on), we find that h_I behaves like the zeroth component of an irreducible tensor of rank one (Shankar, 1980): the relevant angular momentum is here given by the operator $\boldsymbol{\rho}\times\boldsymbol{\pi} + \boldsymbol{\sigma}$. As a consequence, the matrix elements of h_I between states of different j^z are null. More generally, the Wigner–Eckart theorem can be applied; here it quickly adds that $V_{6,10} = V_{10,6} = 0$ (see Section 5). Taking these considerations into account, the first four solutions for $\varepsilon(2)$ are easily obtained from equation (43):

$$\varepsilon(2) = \begin{cases} {}^3S \qquad (2 \text{ roots}) \Leftarrow l=0, \quad s=1, \quad j=1, \quad j^z = \pm 1 & (45) \\ \frac{1}{2}({}^3S + {}^1S) \pm [(B\bar{\mu})^2 + \frac{1}{4}({}^3S - {}^1S)^2]^{1/2} \Leftarrow l=0, & \\ \qquad\qquad s = \frac{1}{2} \pm \frac{1}{2}, \quad j = \frac{1}{2} \pm \frac{1}{2}, \quad j^z = 0 & (46) \end{cases}$$

with

$$\begin{cases} {}^{2s+1}S = W(n=2, l=0, s, j=s) & (47) \\ \bar{\mu} = \left| -\dfrac{1}{B} \langle \mathbf{0}, n=2, l=0, s=0, j=0, j^z=0 | h_I | \mathbf{0}, \right. & \\ \qquad \left. n=2, l=0, s'=1, j'=1, j^z=0 \rangle \right| & \\ = \left| \dfrac{e}{m} \left[\left(1 - \dfrac{5\alpha^2}{96}\right) + a\left(1 - \dfrac{\alpha^2}{96}\right) \right] \right| & (48) \end{cases}$$

The above results are quite similar to the analogous ones for the $n=1$ level (Raspini, 1985); the quantum numbers listed in equations (45) and (46) indicate the unperturbed states to which the calculated corrections refer.

Another three solutions are readily obtained from equation (44), observing that the 10th, 12th, and 16th rows and columns of V are null; this gives

$$\varepsilon(2) = \begin{cases} {}^3P_1 & \Leftarrow \; l=1, \quad s=1, \quad j=1, \quad j^z=0 \qquad (49) \\ {}^3P_2 \text{ (2 roots)} & \Leftarrow \; l=1, \quad s=1, \quad j=2, \quad j^z=\pm 2 \qquad (50) \end{cases}$$

with ${}^{2s+1}P_j = W(n=2, l=1, s, j)$. The remaining nine solutions will come from

$$\det[V_{\gamma\gamma'} - \lambda_\gamma \delta_{\gamma\gamma'}] = 0, \qquad \gamma, \gamma' = 5, \ldots, 9, 11, 13, 14, 15 \qquad (51)$$

Equation (51) has the following structure:

$$\det \left(\begin{array}{c|c|c} \Lambda' & Z & \Delta \\ \hline Z^\dagger & \Lambda'' & N \\ \hline \Delta^\dagger & N & \Lambda''' \end{array} \right) = 0 \qquad (52)$$

where each block is a 3×3 matrix, N is null, and the others are given by

$$\Lambda' = \text{diag}(-\lambda_5, -\lambda_6, -\lambda_7), \qquad \Lambda'' = \text{diag}(-\lambda_8, -\lambda_9, -\lambda_{11}) \qquad (53)$$

$$\Lambda''' = \text{diag}(-\lambda_{13}, -\lambda_{14}, -\lambda_{15}), \qquad \Delta = \text{diag}(V_{5,13}, V_{6,14}, V_{7,15}) \qquad (54)$$

$$Z = \begin{pmatrix} 0 & V_{5,9} & 0 \\ V_{6,8} & 0 & 0 \\ 0 & 0 & V_{7,11} \end{pmatrix} \qquad (55)$$

After some algebra, equation (52) yields

$$[\varepsilon(2) - {}^3P_2][\varepsilon(2) - {}^3P_1][\varepsilon(2) - {}^1P_1] - [\varepsilon(2) - {}^3P_2]|V_{5,9}|^2 - [\varepsilon(2) - {}^3P_1]|V_{5,13}|^2 = 0 \qquad (56)$$

$$[\varepsilon(2) - {}^3P_2][\varepsilon(2) - {}^3P_1][\varepsilon(2) - {}^1P_1] - [\varepsilon(2) - {}^3P_2]|V_{7,11}|^2 - [\varepsilon(2) - {}^3P_1]|V_{7,15}|^2 = 0 \qquad (57)$$

$$[\varepsilon(2) - {}^3P_2][\varepsilon(2) - {}^3P_0][\varepsilon(2) - {}^1P_1] - [\varepsilon(2) - {}^3P_2]|V_{6,8}|^2 - [\varepsilon(2) - {}^3P_0]|V_{6,14}|^2 = 0 \qquad (58)$$

and, due to obvious symmetry reasons [time reversal, Merzbacher (1970)],

$$|V_{5,9}|^2 = |V_{7,11}|^2, \qquad |V_{5,13}|^2 = |V_{7,15}|^2 \qquad (59)$$

so that equations (56) and (57) are actually identical.

Cardan's procedure (Abramowitz, 1970) can be used to solve equation (58). Defining

$$\Sigma = \tfrac{1}{3}[({}^3P_2)+({}^3P_0)+({}^1P_1)] \tag{60}$$

$$\Pi = ({}^3P_2)({}^3P_0)+({}^3P_0)({}^1P_1)+({}^1P_1)({}^3P_2) \tag{61}$$

$$\mathcal{M}^2 = |V_{6,8}|^2/B^2, \qquad \mathcal{N}^2 = |V_{6,14}|^2/B^2 \tag{62}$$

$$\mathcal{P} = (\Pi - 3\Sigma^2) - B^2(\mathcal{M}^2+\mathcal{N}^2) \tag{63}$$

$$\mathcal{Q} = [\Sigma(\Pi-2\Sigma^2)-({}^3P_2)({}^3P_0)({}^1P_1)] - B^2[(\Sigma-{}^3P_2)\mathcal{M}^2+(\Sigma-{}^3P_0)\mathcal{N}^2] \tag{64}$$

$$u = -\frac{\mathcal{Q}}{2}+\left[\left(\frac{\mathcal{P}}{3}\right)^3+\left(\frac{\mathcal{Q}}{2}\right)^2\right]^{1/2} \tag{65}$$

we obtain

$$\varepsilon(2) = \begin{cases} \Sigma+(u'-\mathcal{P}/3u') & \Leftarrow l=1,\ s=0,\ j=1,\ j^z=0 & (66)\\ \Sigma+(u''-\mathcal{P}/3u'') & \Leftarrow l=1,\ s=1,\ j=0,\ j^z=0 & (67)\\ \Sigma+(u'''-\mathcal{P}/3u''') & \Leftarrow l=1,\ s=1,\ j=2,\ j^z=0 & (68) \end{cases}$$

in which u', u'', u''' are the three cube roots of u, in appropriate order.[3] [The square root in equation (65) is a principal value.] Similarly, equation (56) may be treated by means of (60)-(68), with the replacements ${}^3P_0 \to {}^3P_1$, $|V_{6,8}|^2 \to |V_{5,9}|^2$, $|V_{6,14}|^2 \to |V_{5,13}|^2$ as well as $j=0 \to j=1$ [equation (67)] and $j^z=0 \to j^z=-1$ [equations (66)-(68)]; each solution will then be counted twice, to take into account equation (57) ($j^z=1$).

5. MATRIX ELEMENTS: CONCLUSIONS

The typical elements needed to complete the calculations of Section 4 have the structure

$$\langle \mathbf{0}, n=2, l=1, s=0, j=1, j^z | h_I | \mathbf{0}, n=2, l=1, s'=1, j', j^z \rangle \tag{69}$$

and h_I behaves like the zeroth component of an irreducible tensor of rank one. We can define (and, indeed, easily construct) a spherical vector operator $\nu^q_{[1]}$, $q=-1, 0, 1$, such that $\nu^0_{[1]} = h_I$. The Wigner-Eckart theorem then states (Merzbacher, 1970)

$$\begin{aligned} &\langle \mathbf{0}, n=2, l=1, s=0, j=1, j^z | h_I | \mathbf{0}, n=2, l=1, s'=1, j', j^z \rangle \\ &\quad = \frac{3(-1)^{j'-1+j^z}}{\sqrt{3}} \begin{pmatrix} j' & 1 & 1 \\ j^z & 0 & -j^z \end{pmatrix} \\ &\qquad \times \langle \mathbf{0}, n=2, l=1, s=0, j=1 \| \nu_{[1]} \| \mathbf{0}, n=2, l=1, s'=1, j' \rangle \end{aligned} \tag{70}$$

[3]The u-roots are ordered so that the $B \to 0$ limits of solutions (66)-(68) are consistent with the indicated l, s, j quantum numbers; for instance: $\mathrm{Lim}_{B\to 0}$[solution (66)] $= {}^1P_1$.

where $(\cdots)$ is a $3j$-symbol (Messiah, 1966). If $j^z = 0$, the $3j$-symbol vanishes with $j' = 1$: this clearly shows that h_I has a null matrix element between the states characterized by $\gamma = 6$ $(s = 0, j = 1, j^z = 0)$ and $\gamma' = 10$ $(s' = 1, j' = 1, j^z = 0)$. In general, however,

$$|\langle \mathbf{0}, n = 2, l = 1, s = 0, j = 1, j^z | h_I | \mathbf{0}, n = 2, l = 1, s' = 1, j', j^z \rangle| = \left| \frac{Be}{m} \omega_0 \left[\left(1 - \frac{\alpha^2}{\omega_N} \right) + a \left(1 - \frac{\alpha^2}{\omega_A} \right) \right] \right| \tag{71}$$

with the following values of ω_0, ω_N, ω_A:

$$\begin{aligned}
j' = 0, \quad j^z = 0 &\Rightarrow \omega_0 = \sqrt{3}/3, \quad \omega_N = \omega_A = 32 \\
j' = 1, \quad j^z = 0 &\Rightarrow \omega_0 = 0 \\
j' = 1, \quad j^z = \pm 1 &\Rightarrow \omega_0 = \mp\sqrt{2}/2, \quad \omega_N = 16, \quad \omega_A = \infty \\
j' = 2, \quad j^z = 0 &\Rightarrow \omega_0 = -\sqrt{6}/3, \quad \omega_N = 20, \quad \omega_A = 80 \\
j' = 2, \quad j^z = \pm 1 &\Rightarrow \omega_0 = -\sqrt{2}/2, \quad \omega_N = 20, \quad \omega_A = 80
\end{aligned}$$

so that

$$|V_{6,8}|^2 = \frac{B^2\alpha}{3m^2}(1+a)^2 \left(1 - \frac{\alpha^2}{32}\right)^2 = |V_{8,6}|^2 \tag{72}$$

$$|V_{5,9}|^2 = |V_{7,11}|^2 = \frac{B^2\alpha}{2m^2} \left(1 + a - \frac{\alpha^2}{16}\right)^2 = |V_{11,7}|^2 = |V_{9,5}|^2 \tag{73}$$

$$|V_{6,14}|^2 = \frac{2B^2\alpha}{3m^2} \left(1 + a - \frac{\alpha^2}{20} - \frac{a\alpha^2}{80}\right)^2 = |V_{14,6}|^2 \tag{74}$$

$$|V_{5,13}|^2 = |V_{7,15}|^2 = \frac{B^2\alpha}{2m^2} \left(1 + a - \frac{\alpha^2}{20} - \frac{a\alpha^2}{80}\right)^2 = |V_{15,7}|^2 = |V_{13,5}|^2 \tag{75}$$

The results are consistent with those of Grotch and Kashuba (1973), even though the methods employed in this paper are quite different (see also Grotch and Hegstrom, 1971). The procedure we have adopted makes use of relativistic CM variables, which reduce the ten generators of the Poincaré group (for the isolated system) to the single-particle form at this order of approximation (Krajcik and Foldy, 1974). Specifically, the use of relativistic CM variables produces terms in the h_I operator which would not be present otherwise: therefore, with nonrelativistic CM variables, one would have to modify the unperturbed states in order to achieve the same precision (Raspini, 1985).

REFERENCES

Abramowitz, M. (1970). In *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Washington, D.C.
Barker, W. A., and Glover, F. N. (1955). *Physical Review*, **99**, 317.
Breit, G. (1947). *Physical Review*, **71**, 400.
Close, F. E., and Osborn, H. (1970). *Physical Review D*, **2**, 2127.
Fulton, T., and Martin, P. C. (1954). *Physical Review*, **93**, 903.
Grotch, H., and Hegstrom, R. A. (1971). *Physical Review A*, **4**, 59.
Grotch, H., and Kashuba, R. (1973). *Physical Review A*, **7**, 78.
Karplus, R., and Klein, A. (1952). *Physical Review*, **87**, 848.
Krajcik, R. A., and Foldy, L. L. (1974). *Physical Review D*, **10**, 1777.
Merzbacher, E. (1970). *Quantum Mechanics*, Wiley, New York.
Messiah, A. (1966). *Quantum Mechanics*, Wiley, New York.
Raspini, A. (1985). *Journal of Physics B*, **18**, 3859.
Sakurai, J. J. (1967). *Advanced Quantum Mechanics*, Addison-Wesley, Reading, Massachusetts.
Sebastian, K. J. (1979). *Physical Review A*, **19**, 1398.
Sebastian, K. J. (1981). *Physical Review A*, **23**, 2810.
Sebastian, K. J., and Yun, D. (1979). *Physical Review D*, **19**, 2509.
Shankar, R. (1980). *Principles of Quantum Mechanics*, Plenum Press, New York.